Recitation 11

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Problems

Problem 1. First you need to find basis for W. So just solve the system of equations. Row reducing, you get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

So the variable x_3 is free, and solutions are of the form $u = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, i.e. $W = Span(u)$. The set $\{u\}$ is orthogonal by stupid reasons: it only has one vector. So we have an orthogonal basis of W, and we can use the projection formula. We have

$$\widehat{y} = proj_W(y) = \frac{-3 \cdot (-1) + 2 \cdot 1 + 4 \cdot 1}{(-1)^2 + 1^2 + 1^2} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$$

Then the distance from y to W is the length $||y - \widehat{y}|| = ||[0, 1, -1]^T|| = \sqrt{2}$. The projection $\widehat{y}$ is the closest point to y inside W.

Problem 2. Method 1. Find a basis of W. Solving the system of equations $x_1 - 3x_2 - x_3 = 0$ you get that the basis of W is given by the vectors

$$x_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This basis is not orthogonal, so we can't use the projection formula right away. We can use Gramm-Schmidt algorithm to orthogonalize our basis. It is more convenient to start the algorithm with the second vector rather than the first, since the former has smaller length.

So we have $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and then

$$v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} - \frac{3 \cdot 1 + 1 \cdot 0 + 0 \cdot 1}{1^2 + 0^2 + 1^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ -3/2 \end{bmatrix}$$

It is easy to see that the result is indeed orthogonal. Now we can use the projection formula to get

$$proj_W(y_1) = \frac{2 \cdot 1 + (-1) \cdot 0 + (-3) \cdot 1}{1^2 + 0^2 + 1^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{2 \cdot (3/2) + (-1) \cdot 1 + (-3) \cdot (-3/2)}{(3/2)^2 + 1^2 + (-3/2)^2} \begin{bmatrix} 3/2 \\ 1 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 14/11 \\ 13/11 \\ -25/11 \end{bmatrix}.$$

Method 2. Any vector y can be written uniquely as $y = \widehat{y} + r$, where $\widehat{y} = proj_W(y)$ is the projection, and r is perpendicular to W. If you know $\widehat{y}$, you can find r, and vice versa. Here it is actually faster to find r. This vector r would be just the projection of y onto W^T, and W^T is easy to describe: it is spanned by the vector $[1, -3, -1]^T$ (notice that this is exactly the vector y_2 I asked you to project). Indeed, W was defined as all the vectors $[x_1, x_2, x_3]^T$ satisfying the equation $x_1 - 3x_2 - x_3 = 0$. But this equation is exactly the condition of $[x_1, x_2, x_3]^T$ being perpendicular to $[1, -3, -1]^T$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = x_1 - 3x_2 - x_3 = 0$$

Thus the projection r of y onto $W^\perp$ is given by

$$r = \frac{2 \cdot 1 + (-1) \cdot (-3) + (-3) \cdot (-1)}{1^2 + (-3)^2 + (-1)^2} \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/11 \\ -24/11 \\ -8/11 \end{bmatrix}$$

Then the projection $\widehat{y}_1 = proj_W(y_1)$ will be $y_1 - r$, i.e.

$$proj_W(y_1) = y_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} - \begin{bmatrix} 8/11 \\ -24/11 \\ -8/11 \end{bmatrix} = \begin{bmatrix} 14/11 \\ 13/11 \\ -25/11 \end{bmatrix}$$

Notice that this is exactly the same result that we've got from the first method.

OK, I agree, this is pretty ugly. I should have checked the calculations before giving assigning you the problem.

But it would be much nicer for the projections of y_2 and y_3. First of all, the vector y_2 is perpendicular to W (see the discussion in Method 2). So the projection of y_2 onto W is just 0. Speaking of y_3, you can check that y_3 satisfies the equation $x_1 - 3x_2 - x_3 = 0$, and so $y_3 \in W$, and therefore $proj_W(y_3) = y_3$.

Problem 3. The calculations in this problem are even uglier than in the previous one (in fact, much uglier). They are so ugly, I couldn't even finish them. So, I am sorry, no answer for this problem. The main points are, though, you first need to apply Gramm-Schmidt algorithm. It will give you orthogonal vectors v_1, v_2, v_3. Then you will have to normalize them, and this would be the end. There is **no point** in normalizing the vectors **before** applying Gramm-Schmidt.

Problem 4. Since I couldn't compute the previous problem, I can't compute this one either. Sad...
But the main point is, the resulting orthogonal vectors v_1, v_2, v_3 you and I were supposed to obtain from Gramm-Schmidt algorithm in the previous exercise will give the columns of the matrix $Q = [v_1 v_2 v_3]$. Then R would be found by $R = Q^T A$.

Problem 5. Knowing Q from the previous exercise, we would compute $proj_W(y) = UU^T y$. Since we don't know it, we have to move on.

Problem 6. This one is obvious. Look at the formulas ans see that all the dot products $x_i \cdot v_j$ in the formula will be zero. For example, when computing v_2, $x_2 \cdot v_1 = 0$, and so $v_2 = x_2$, and so on.

Problem 7. Let $W \subset \mathbb{R}^3$ be a plane $x_1 - x_2 + x_3 = 0$, and let $T\colon \mathbb{R}^3 \to \mathbb{R}^{\nVdash}$ be the transformation $v \mapsto T(v) := proj_W(v)$.

- Without doing any calculation, explain why $T \circ T = T$, i.e. applying T twice is the same as applying it once.
- Find a basis of W. Call it $\{v_1, v_2\}$.
- Let $v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Explain why $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
- Find the matrix of T relative to this basis.
- Analyze what you've got.

First, for any vector $x \in \mathbb{R}^3$, $T(x)$ is inside W, and so if we try to project it on W again, it will just stay where it wa, since it is already in W. But that exactly means that $T(T(x)) = T(x)$, for any x. So $T^2 = T$.

I hope you all know how to find a basis of a subspace W given by equation $x_1 - x_2 + x_3 = 0$. If you don't know that, please, see me during my office hours ASAP. So, we can take

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The set $\mathcal{B} = \{v_1, v_2, v_3\}$ will be linearly independent since v_3 is non-zero, and orthogonal to $W = Span(v_1, v_2)$. Since we are in $\mathbb{R}^3$, any three vectors form a basis, so we are good.

Since v_1, v_2 are in W, T doesn't do anything to them, and so $T(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$ and similarly $T(v_2) = v_2 = 0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3$. Since v_3 is perpendicular to W, we get $T(v_3) = 0$. Thus the matrix is just

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 8. Let A be an $m \times n$ matrix. Prove that any vector x in $\mathbb{R}^n$ can be written uniquely as a sum $x = p + u$, where $p \in Row(A)$ and $u \in Nul(A)$. (Hint: what is $Row(A)^{\perp}$?)
Since I think it is a good problem to think about, I will just briefly sketch the solution. Check using the definition (and using how matrix multiplication works) that $Row(A)^{\perp}$ is exactly the null space $Nul(A)$. Than use Theorem 8 in section 6.3.

Now, write $x = p + u$ as in the first part of the problem. What happens if you apply A to $p + u$?